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Hypersonic strong-interaction similarity solutions for flow past a flat plate

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The hypersonic strong-interaction regime for the flow of a viscous, heat-conducting compressible fluid past a flat plate is analysed using the Navier–Stokes equations as a basis. It is assumed that the fluid is a perfect gas having constant specific heats, a constant Prandtl number σ, whose numerical value is of order one, and a viscosity coefficient varying as a power, ω, of the absolute temperature. Limiting forms of solutions are studied as the free-stream Mach number M, the free-stream Reynolds number based on the plate length R_L, and the interaction parameter $\chi = \{(\gamma M^2)^{2+\omega}/R_L\}^{\frac{1}{2}}$, go to infinity.

Through the use of asymptotic expansions and matching, it is shown that, for $(1-\omega) > 0$, three distinct layers for which similarity exists make up the region between the shock wave and the plate. The behaviour of the flow in these three layers is analysed.

1. Introduction

The purpose of this paper is to enlarge upon the existing theory for the hypersonic strong-interaction problem for viscous, compressible flow past a flat plate (cf. e.g. Stewartson 1964).

According to this strong-interaction theory, the flow field is divided into three regions: (*a*) a region extending from the upstream side of a Rankine–Hugoniot shock wave outwards, where there is a uniform, high-speed flow; (*b*) a high temperature, low density, viscous region extending outward from the plate part of the way to the shock, across which the pressure change is small so that the flow in this region can be analysed by boundary-layer theory; and (*c*) an inviscid region, between the clearly defined outer edge of the above viscous boundary layer and the downstream side of the Rankine–Hugoniot shock wave, for which the hypersonic small-disturbance theory holds.

The Stewartson solution of this flow problem for $\omega = 1$ consists of a similar solution for the flow in the inviscid shock layer that is joined to the similar solution for the flow in the viscous boundary layer. Stewartson states, however, that to make this joining, a slight change in the boundary conditions at the outer edge of the viscous layer has to be made. It is this need for modification in these boundary conditions that drew the author's attention to the strong-interaction problem. A standard method for overcoming such a difficulty is the introduction of a layer intermediate to the shock and boundary layers. For $\omega = 1$, unfortunately, the intermediate layer is not the answer to the problem. For $(1-\omega) > 0$, however, it is found that, with an intermediate layer, strong-interaction similarity solutions, which do not require modification of the boundary conditions at the outer edge of the boundary layer, can be obtained. These similarity solutions for $(1-\omega) > 0$ are discussed in the following sections.

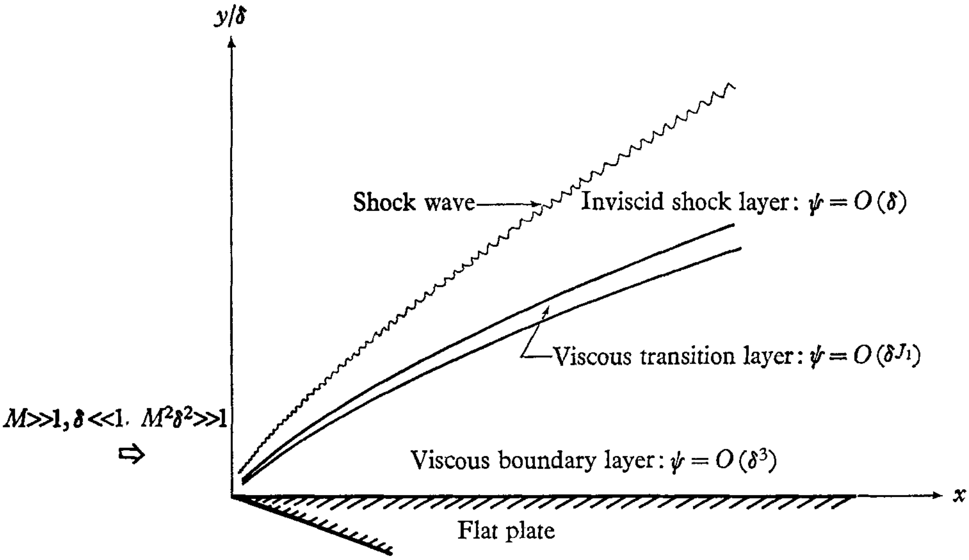

FIGURE 1. Schematic diagram of hypersonic strong interaction for flow past a flat plate.

2. The equations of motion

Consider the (two-dimensional) flow of a viscous, compressible gas past a semi-infinite flat plate. Let $x_1 = Lx$ and $y_1 = Ly$ represent the Cartesian co-ordinates parallel and normal to the flat plate, respectively, with the origin of this co-ordinate system at the leading edge of the plate. The length L is chosen so that x is of order unity in the region where the strong-interaction theory is valid. The velocity components in the x_1- and y_1-directions are $u_1 = u_\infty u$, and $v_1 = u_\infty v$, and the pressure, temperature, and density are $p_1 = p_\infty p$, $T_1 = T_\infty T$, and $\rho_1 = \rho_\infty \rho$, where u_∞, p_∞, T_∞, and ρ_∞ are the velocity in the x_1-direction, pressure, temperature, and density in the undisturbed region upstream of the flat plate.

The gas is assumed to be a perfect one ($p = \rho T$), having (i) constant specific heats, c_{v_1} and c_{p_1}, with $\gamma = (c_{p_1}/c_{v_1})$, such that $(\gamma-1) = O(1)$, (ii) a constant Prandtl number of order unity ($\sigma = \text{const.} = O(1)$), and (iii) its 'normal' viscosity coefficient proportional to a power, ω, of the absolute temperature ($\mu_1 = \mu_\infty \mu = \mu_\infty T^\omega$, with $\frac{1}{2} \leqslant \omega < 1$, as will be shown to be required in the succeeding analysis), while its 'bulk' viscosity coefficient is taken to be zero, although such an assumption is not necessary.

The von Mises forms of the Navier–Stokes equations for the flow of such a gas are

$$\frac{\partial}{\partial \psi}\left(\frac{v}{u}\right) - \frac{\partial}{\partial \xi}\left(\frac{1}{\rho u}\right) = 0, \tag{2.01}$$

$$\rho u \frac{\partial u}{\partial \xi} + \frac{1}{\gamma M^2}\left(\frac{\partial p}{\partial \xi} - \rho v \frac{\partial p}{\partial \psi}\right)$$
$$= \frac{1}{R_L}\left[\!\!\left[\rho u \frac{\partial}{\partial \psi}\left[T^\omega\left\{\rho u \frac{\partial u}{\partial \psi} + \left(\frac{\partial v}{\partial \xi} - \rho v \frac{\partial v}{\partial \psi}\right)\right\}\right] + \left(\frac{\partial}{\partial \xi} - \rho v \frac{\partial}{\partial \psi}\right)\right.\right.$$
$$\left.\left.\times\left[T^\omega\left\{\frac{4}{3}\left(\frac{\partial u}{\partial \xi} - \rho v \frac{\partial u}{\partial \psi}\right) - \tfrac{2}{3}\rho u \frac{\partial v}{\partial \psi}\right\}\right]\right]\!\!\right], \quad (2.02)$$

$$\rho u\left(\frac{\partial v}{\partial \xi} + \frac{1}{\gamma M^2}\frac{\partial p}{\partial \psi}\right)$$
$$= \frac{1}{R_L}\left[\!\!\left[\rho u \frac{\partial}{\partial \psi}\left[T^\omega\left\{\tfrac{4}{3}\rho u \frac{\partial v}{\partial \psi} - \frac{2}{3}\left(\frac{\partial u}{\partial \xi} - \rho v \frac{\partial u}{\partial \psi}\right)\right\}\right] + \left(\frac{\partial}{\partial \xi} - \rho v \frac{\partial}{\partial \psi}\right)\right.\right.$$
$$\left.\left.\times\left[T^\omega\left\{\rho u \frac{\partial u}{\partial \psi} + \left(\frac{\partial v}{\partial \xi} - \rho u \frac{\partial v}{\partial \psi}\right)\right\}\right]\right]\!\!\right], \quad (2.03)$$

$$\rho u \frac{\partial T}{\partial \xi} - \left(\frac{\gamma - 1}{\gamma}\right) u \frac{\partial p}{\partial \xi}$$
$$= \frac{1}{\sigma R_L}\left[\!\!\left[\rho u \frac{\partial}{\partial \psi}\left(T^\omega \rho u \frac{\partial T}{\partial \psi}\right) + \left(\frac{\partial}{\partial \xi} - \rho v \frac{\partial}{\partial \psi}\right)\left(T^\omega\left\{\frac{\partial T}{\partial \xi} - \rho v \frac{\partial T}{\partial \psi}\right\}\right)\right]\!\!\right]$$
$$+ \left(\frac{\gamma - 1}{\gamma}\right)\frac{\gamma M^2}{R_L} T^\omega\left[\!\!\left[\left\{\rho u \frac{\partial u}{\partial \psi} + \left(\frac{\partial v}{\partial \xi} - \rho v \frac{\partial v}{\partial \psi}\right)\right\}^2 + 2\left\{\left(\frac{\partial u}{\partial \xi} - \rho v \frac{\partial u}{\partial \psi}\right)^2 + \left(\rho u \frac{\partial v}{\partial \psi}\right)^2\right\}\right.\right.$$
$$\left.\left. - \frac{2}{3}\left\{\left(\frac{\partial u}{\partial \xi} - \rho v \frac{\partial u}{\partial \psi}\right) + \rho u \frac{\partial v}{\partial \psi}\right\}^2\right]\!\!\right], \quad (2.04)$$

where $\xi = x$ and ψ is the stream function, defined by

$$(\partial \psi / \partial y) = \rho u, \quad (\partial \psi / \partial x) = -\rho v;$$

$M^2 = (\rho_\infty u_\infty^2 / \gamma p_\infty)$, the square of the free-stream Mach number, and $R_L = (\rho_\infty u_\infty L / \mu_\infty)$, the Reynolds number. The analysis presented here is for $M^2 \gg 1$ and $R_L \gg 1$.

3. The inviscid shock layer

According to the existing hypersonic strong-interaction theory for a flat plate, at the surface there is a thin, viscous, heat-conducting layer, which disturbs the external flow. This layer, described by $y = \delta Y_b(x)$, with δ, the thickness parameter, much less than unity, acts as an effective slender 'body', producing an oblique shock wave, $y = \delta Y_{sh}(x) > \delta Y_b(x)$, and an inviscid shock layer between the shock wave and the 'body'. This inviscid shock layer, in which

$$\delta Y_b(x) \leqslant y \leqslant \delta Y_{sh}(x),$$

satisfies the hypersonic small-disturbance-theory equations. Just such an inviscid shock layer provides the starting point for the present analysis.

For the inviscid shock layer, in accordance with the work of Van Dyke (1954), the expansions of the flow variables are carried out in the distorted von Mises co-ordinates

$$\xi_h = \xi, \quad \psi_h = \psi / \delta \quad (3.01)$$

and have the form

$$\left.\begin{aligned} u &= 1+\delta^2 u_h(\xi_h, \psi_h)+\dots, \\ v &= \delta v_h(\xi_h, \psi_h)+\dots, \\ p &= \gamma M^2\delta^2 p_h(\xi_h, \psi_h)+\dots, \\ T &= \gamma M^2\delta^2 T_h(\xi_h, \psi_h)+\dots, \\ \rho &= \rho_h(\xi_h, \psi_h)+\dots. \end{aligned}\right\} \tag{3.02}$$

Thus, the first-approximation equations of hypersonic small-disturbance theory† under the von Mises transformation are

$$\frac{\partial v_h}{\partial \psi_h}-\frac{\partial(1/\rho_h)}{\partial \xi_h}=0, \tag{3.03}$$

$$\rho_h\frac{\partial u_h}{\partial \xi_h}+\left(\frac{\partial p_h}{\partial \xi_h}-\rho_h v_h\frac{\partial p_h}{\partial \psi_h}\right)=0, \tag{3.04}$$

$$\frac{\partial v_h}{\partial \xi_h}+\frac{\partial p_h}{\partial \psi_h}=0, \tag{3.05}$$

$$\rho_h\frac{\partial T_h}{\partial \xi_h}-\left(\frac{\gamma-1}{\gamma}\right)\frac{\partial p_h}{\partial \xi_h}=0. \tag{3.06}$$

The Rankine–Hugoniot oblique shock relations determine the boundary conditions for the flow quantities of equation (3.02) at

$$(y_h)_{sh}=Y_{sh}(x_h), \quad \text{or} \quad (\psi_h)_{sh}=\Psi_{sh}(\xi_h)$$

(where, for the two-dimensional flow, $(\psi_h)_{sh}=(y_h)_{sh}$). These shock relations in the distorted von Mises co-ordinates are

$$\left.\begin{aligned} &(p_h)_{sh}=\frac{2}{\gamma+1}[\Psi'_{sh}(\xi_h)]^2, \quad (\rho_h)_{sh}=\frac{\gamma+1}{\gamma-1}, \quad (T_h)_{sh}=\frac{2(\gamma-1)}{(\gamma+1)^2}[\Psi'_{sh}(\xi_h)]^2, \\ &(u_h)_{sh}=\frac{-2}{\gamma+1}[\Psi'_{sh}(\xi_h)]^2, \quad (v_h)_{sh}=\frac{2}{\gamma+1}\Psi'_{sh}(\xi_h), \end{aligned}\right\} \tag{3.07}$$

taking $M^2\delta^2 \gg 1$.

Looking for similar solutions to the overall flat-plate problem, it is necessary to consider similar solutions for the inviscid shock layer. For $M^2\delta^2 \gg 1$, it is known that the flow in the inviscid shock layer is self-similar if the shock associated with this flow is a power-law shock, i.e. $(y_h)_{sh} = x_h^n$‡ or $(\psi_h)_{sh} = \xi_h^n$. For such a solution, the independent variables are

$$\xi_h \quad \text{and} \quad \zeta_h = (\psi_h/\xi_h^n), \tag{3.08}$$

† The ratios of the orders of magnitude of the leading viscosity and heat-conduction terms, which have been neglected, to those of the inviscid terms, which have been retained, is $\{(\gamma M^2)^\omega/R_L\delta^4\}.\delta^{2(1+\omega)}$.

‡ Note that in physical co-ordinates the equation for the shock shape is $(y_1)_{sh} = Wx_1^n$, where W is a shape constant. This means that $(y)_{sh} = (W/L^{1-n})x^n$, so that the quantity δ for the power-law shock is $\delta = (W/L^{1-n})$.

so that $(\zeta_h)_{sh} = 1$. The shock relations indicate that the dependent variables should be expressed as

$$\left.\begin{aligned} p_h &= \xi_h^{-2(1-n)} P_h(\zeta_h), \quad T_h = \xi_h^{-2(1-n)} \Theta_h(\zeta_h),\\ \rho_h &= D_h(\zeta_h),\\ u_h &= \xi_h^{-2(1-n)} U_h(\zeta_h), \quad v_h = \xi_h^{-(1-n)} V_h(\zeta_h). \end{aligned}\right\} \tag{3.09}$$

The equations of motion, with the introduction of equations (3.08) and (3.09), reduce to

$$\left.\begin{aligned} &\frac{dV_h}{d\zeta_h} - \frac{n\zeta_h}{D_h^2}\frac{dD_h}{d\zeta_h} = 0, \quad P_h = D_h\Theta_h,\\ &D_h\left[2(1-n)U_h + n\zeta_h\frac{dU_h}{d\zeta_h}\right] + \left[2(1-n)P_h + n\zeta_h\frac{dP_h}{d\zeta_h}\right] + D_h V_h\frac{dP_h}{d\zeta_h} = 0,\\ &\left[(1-n)V_h + n\zeta_h\frac{dV_h}{d\zeta_h}\right] = \frac{dP_h}{d\zeta_h},\\ &D_h\left[2(1-n)\Theta_h + n\zeta_h\frac{d\Theta_h}{d\zeta_h}\right] = \left(\frac{\gamma-1}{\gamma}\right)\left[2(1-n)P_h + n\zeta_h\frac{dP_h}{d\zeta_h}\right]. \end{aligned}\right\} \tag{3.10}$$

The boundary conditions for these equations, from the shock relations, are

$$\left.\begin{aligned} &P_h(1) = 2n^2/(\gamma+1), \quad D_h(1) = (\gamma+1)/(\gamma-1), \quad \Theta_h(1) = 2(\gamma-1)n^2/(\gamma+1)^2,\\ &U_h(1) = -2n^2/(\gamma+1), \quad V_h(1) = 2n/(\gamma+1). \end{aligned}\right\} \tag{3.11}$$

For $\frac{2}{3} < n \leqslant 1$, the solution of equations (3.10) and (3.11) yields, as $\zeta_h \to 0$,

$$p_h = \xi_h^{-2(1-n)} P_0 + \ldots, \quad v_h = \xi_h^{-(1-n)} V_0 + \ldots, \tag{3.12}$$

where P_0 and V_0 are constants, whose values depend upon the values of γ and n. The solutions for ρ_h and T_h, as $\zeta_h \to 0$ (or $\psi_h \to 0$, keeping ξ_h fixed), are

$$\left.\begin{aligned} \rho_h &= [\{(\gamma+1)/(\gamma-1)\}\{(\gamma+1)P_0/2n^2\}^{1/\gamma}]\,\zeta_h^{2(1-n)/n\gamma} + \ldots\\ &= D_0(\psi_h/\xi_h^n)^{2(1-n)/n\gamma} + \ldots,\\ T_h &= (P_0/D_0)\,\xi_h^{-2(1-n)}\,\zeta_h^{-2(1-n)/n\gamma} + \ldots\\ &= \Theta_0\xi_h^{-2(1-n)(\gamma-1)/\gamma}\psi_h^{-2(1-n)/n\gamma} + \ldots, \end{aligned}\right\} \tag{3.13}$$

where D_0 and Θ_0 are constants that are known once the values of γ and n are prescribed. From equation (3.13), it is clear that one or more layers interior to the shock layer must be introduced in which $\rho \ll O(1)$ and $T \gg O(\gamma M^2\delta^2)$, in order to remove the above singular behaviour as $\psi_h \to 0$.

In the next two sections, the two interior layers that are necessary to remove this singular behaviour and to complete the description of the flow are described.

4. The viscous boundary layer

Consider next a viscous boundary layer, a high-temperature, low-density region, across which the pressure is constant (i.e. p is a function of ξ alone), and at the outer edge of which the flow quantities u, T, v, and p have the behaviour: $u \to 1$, $(T/\gamma M^2) \to 0$, $v \to \delta\xi^{-(1-n)}V_0$, and $p \to \gamma M^2\delta^2\xi^{-2(1-n)}P_0$, where V_0 and P_0 are the constants introduced in equation (3.12). This is essentially the viscous layer studied by Stewartson (1964), *et al.*

The expansions for this region are carried out in the distorted co-ordinates

$$\xi_b = \xi, \quad \psi_b = \psi/\delta^3 \tag{4.01}$$

and have the form

$$\left.\begin{aligned} u &= u_b + \ldots, \quad v = \delta v_b + \ldots, \quad p = \gamma M^2 \delta^2 p_b + \ldots, \\ T &= \gamma M^2 T_b + \ldots, \quad \rho = \delta^2 \rho_b + \ldots . \end{aligned}\right\} \tag{4.02}$$

These are also of the same orders of magnitude as for the viscous layer in the usual strong-interaction theory.

For these expansions, the leading terms in the equations of motion are

$$\left.\begin{aligned} &\frac{\partial}{\partial \psi_b}\left(\frac{v_b}{u_b}\right) = \frac{\partial}{\partial \xi_b}\left(\frac{T_b}{p_b u_b}\right), \quad \frac{\partial p_b}{\partial \psi_b} = 0, \\ &u_b \frac{\partial u_b}{\partial \xi_b} + \frac{1}{p_b}\frac{dp_b}{d\xi_b} T_b = \left\{\frac{(\gamma M^2)^\omega}{R_L \delta^4}\right\} p_b u_b \frac{\partial}{\partial \psi_b}\left(\frac{u_b}{T_b^{1-\omega}}\frac{\partial u_b}{\partial \psi_b}\right), \\ &\frac{\partial T_b}{\partial \xi_b} - \left(\frac{\gamma-1}{\gamma}\right)\frac{1}{p_b}\frac{dp_b}{d\xi_b} T_b \\ &\quad = \left\{\frac{(\gamma M^2)^\omega}{R_L \delta^4}\right\} p_b \left[\frac{1}{\sigma}\frac{\partial}{\partial \psi_b}\left(\frac{u_b}{T_b^{1-\omega}}\frac{\partial T_b}{\partial \psi_b}\right) + \left(\frac{\gamma-1}{\gamma}\right)\frac{u_b}{T_b^{1-\omega}}\left(\frac{\partial u_b}{\partial \psi_b}\right)^2\right]. \end{aligned}\right\} \tag{4.03}$$

To retain the viscosity and heat-conduction terms, it is necessary that the quantity $\{(\gamma M^2)^\omega / R_L \delta^4\} \equiv \Lambda$ be of $O(1)$, so that

$$\delta = [(\gamma M^2)^\omega / R_L \Lambda]^{\frac{1}{4}} \to 0. \tag{4.04}$$

Combining equation (4.04) with the inequality $\gamma M^2 \delta^2 \gg 1$ yields

$$\{(\gamma M^2)^{2+\omega}/R_L\}^{\frac{1}{2}} \gg 1,$$

which is the generalization of the usual criterion for strong interaction: that the interaction parameter, $\chi = M^3/R_L^{\frac{1}{2}}$ for $\omega = 1$, be much greater than $O(1)$. Further, since $\delta \ll 1$ and $\gamma M^2 \delta^2 \gg 1$, the range of the order of magnitude of R_L is $M^{2\omega} \ll R_L \ll M^{2(2+\omega)}$. Since L is the measure of x_1, it follows that the results should be valid for $M^{2\omega}(\mu_\infty/\rho_\infty u_\infty) \ll x_1 \ll M^{2(2+\omega)}(\mu_\infty/\rho_\infty u_\infty)$.

The above equations (4.03), satisfying the boundary conditions at the outer edge

$$u_b \to 1, \quad T_b \to 0, \quad v_b \to \xi_b^{-(1-n)} V_0, \quad p_b \to \xi_b^{-2(1-n)} P_0 \quad \text{as} \quad \psi_b \to \infty, \tag{4.05}$$

may be reduced to ordinary differential equations if

$$n = \tfrac{3}{4}. \tag{4.06}$$

Therefore, for $n = \frac{3}{4}$, taking the independent variables to be

$$\xi_b \quad \text{and} \quad \zeta_b = \psi_b / \xi_b^{\frac{1}{4}}, \tag{4.07}$$

and taking the dependent variables to be

$$\left.\begin{aligned} u_b &= U_b(\zeta_b), \quad v_b = \xi_b^{-(1-n)} V_b(\zeta_b) = \xi_b^{-\frac{1}{4}} V_b(\zeta_b), \\ T_b &= \Theta_b(\zeta_b), \quad p_b = \xi_b^{-2(1-n)} P_0 = \xi_b^{-\frac{1}{2}} P_0, \end{aligned}\right\} \tag{4.08}$$

the continuity, momentum, and energy equations become

$$\left.\begin{aligned}\frac{d}{d\zeta_b}\left(\frac{V_b}{U_b}\right)+\frac{1}{P_0}\left[\tfrac{1}{4}\zeta_b\frac{d}{d\zeta_b}\left(\frac{\Theta_b}{U_b}\right)-\frac{1}{2}\left(\frac{\Theta_b}{U_b}\right)\right] &= 0,\\ \Lambda P_0 U_b\frac{d}{d\zeta_b}\left(\frac{U_b}{\Theta_b^{1-\omega}}\frac{dU_b}{d\zeta_b}\right)+\tfrac{1}{4}\zeta_b U_b\frac{dU_b}{d\zeta_b}+\tfrac{1}{2}\Theta_b &= 0,\\ \Lambda P_0\left[\frac{1}{\sigma}\frac{d}{d\zeta_b}\left(\frac{U_b}{\Theta_b^{1-\omega}}\frac{d\Theta_b}{d\zeta_b}\right)+\left(\frac{\gamma-1}{\gamma}\right)\frac{U_b}{\Theta_b^{1-\omega}}\left(\frac{dU_b}{d\zeta_b}\right)^2\right]+\tfrac{1}{4}\zeta_b\frac{d\Theta_b}{d\zeta_b}-\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\Theta_b &= 0.\end{aligned}\right\} \quad (4.09)$$

The boundary conditions for these equations at the outer edge and at the wall are

$$\left.\begin{aligned}&U_b\to 1,\quad \Theta_b\to 0,\quad V_b\to V_0,\quad \text{as}\quad \zeta_b\to\infty,\\ &U_b\to 0,\quad \Theta_b\to\Theta_{b,w}\neq 0,\quad V_b\to 0,\quad \text{as}\quad \zeta_b\to 0.\end{aligned}\right\} \quad (4.10)$$

As yet no requirements have been set for the way in which U_b, Θ_b, and V_b approach these conditions as $\zeta_b\to\infty$ and $\zeta_b\to 0$. A discussion of the solutions of equations (4.09) and (4.10) is presented in the Appendix.

Since $\Theta_b\to 0$ as $\zeta_b\to\infty$, in order to match with the colder inviscid shock layer, and $U_b\to 1$ as $\zeta_b\to\infty$, consider the following asymptotic expansions for Θ_b and U_b as $\zeta_b\to\infty$

$$\left.\begin{aligned}\Theta_b &\sim a_1\zeta_b^{-\alpha_1}+a_2\zeta_b^{-\alpha_2}+\ldots; \qquad 0<\alpha_1<\alpha_2<\ldots,\\ U_b &\sim 1+b_1\zeta_b^{-\beta_1}+b_2\zeta_b^{-\beta_2}+\ldots; \quad 0<\beta_1<\beta_2<\ldots.\end{aligned}\right\} \quad (4.11)$$

Substitution of these expansions into the momentum and energy equations of (4.09) produces the results that, as $\zeta_b\to\infty$, for $(1-\omega)>0$,

$$\left.\begin{aligned}a_1 &= \left[\frac{1+\omega}{1-\omega}\frac{4\Lambda P_0/\sigma}{\{1+(1-\omega)(\gamma-1)/\gamma\}}\right]^{1/(1-\omega)}, \quad \alpha_1=\frac{2}{1-\omega},\\ b_1 &= a_1\left[\frac{1-\omega}{1-\sigma\{1+(1-\omega)(\gamma-1)/\gamma\}}\right], \quad \beta_1=\alpha_1=\frac{2}{1-\omega}.\end{aligned}\right\} \quad (4.12)$$

The quantity a_1 is positive (for $(1-\omega)>0$). The quantity b_1 is also positive (for $(1-\omega)>0$), when $(1/\sigma)$ is greater than $\{1+(1-\omega)(\gamma-1)/\gamma\}$. A realistic value for the quantity $(1/\sigma)$ is that given by Eucken (1913), being

$$(1/\sigma)=\{1+\tfrac{5}{4}(\gamma-1)/\gamma\}>\{1+(1-\omega)(\gamma-1)/\gamma\}.$$

Therefore, the temperature T_b near the outer edge of the viscous boundary layer ($\psi_b\to\infty$, $\xi=\xi_b$ fixed) is

$$T_b\sim a_1\xi_b^{1/2(1-\omega)}\psi_b^{-2/(1-\omega)}+\ldots\,. \quad (4.13)$$

From a comparison of equations (3.13) and (4.13), it is clear that the functional behaviour of the temperature in the inviscid shock layer and the viscous boundary layer as $\psi_h\to 0$ and $\psi_b\to\infty$, respectively, does not permit direct matching, and that a transition layer intermediate to these layers must be introduced in order that there may be matching.

5. The viscous transition layer

To span the distance between the inviscid shock layer and the viscous boundary layer, a viscous transition layer is introduced. Subject to verification by matching of this region to both of the adjacent regions, the distorted co-ordinates and the expansions for this transition layer are taken to be

$$\xi_t = \xi, \quad \psi_t = \psi/\phi_t, \quad \delta^3 \ll O(\phi_t) \ll \delta. \tag{5.01}$$

$$\left.\begin{aligned} u &= 1+\theta_t u_t+\ldots, \quad \delta^2 \ll O(\theta_t) \ll 1, \\ v &= \delta v_t+\ldots, \quad p = \gamma M^2\delta^2 p_t+\ldots, \\ T &= \gamma M^2\theta_t T_t+\ldots, \quad \rho = (\delta^2/\theta_t)\rho_t+\ldots,\end{aligned}\right\} \tag{5.02}$$

with the parameter $\theta_t\phi_t/\delta^3$ taken to be approaching zero.

The leading terms in the equations of motion for this layer are

$$\left.\begin{aligned} &\frac{\partial v_t}{\partial\psi_t} = O\left(\frac{\theta_t\phi_t}{\delta^3}\right)\to 0, \quad \frac{\partial p_t}{\partial\psi_t} = O\left(\frac{\phi_t}{\delta}\right)\to 0, \\ &\frac{\partial u_t}{\partial\xi_t}+\frac{1}{p_t}\frac{\partial p_t}{\partial\xi_t}T_t = \left\{\frac{(\gamma M^2)^\omega}{R_L\delta^4}\right\}\left\{\frac{\delta^6}{\phi_t^2\theta_t^{1-\omega}}\right\}p_t\frac{\partial}{\partial\psi_t}\left(\frac{1}{T_t^{1-\omega}}\frac{\partial u_t}{\partial\psi_t}\right), \\ &\frac{\partial T_t}{\partial\xi_t}-\left(\frac{\gamma-1}{\gamma}\right)\frac{1}{p_t}\frac{\partial p_t}{\partial\xi_t}T_t = \frac{1}{\sigma}\left\{\frac{(\gamma M^2)^\omega}{R_L\delta^4}\right\}\left\{\frac{\delta^6}{\phi_t^2\theta_t^{1-\omega}}\right\}p_t\frac{\partial}{\partial\psi_t}\left(\frac{1}{T_t^{1-\omega}}\frac{\partial T_t}{\partial\psi_t}\right).\end{aligned}\right\} \tag{5.03}$$

From this equation, it can be seen that the normal velocity v_t and the pressure p are constant across this transition layer and are the values of these quantities at the inner edge of the inviscid shock layer ($\psi_h\to 0$). Therefore, for $n = \frac{3}{4}$,

$$v_t = \xi_t^{-\frac{1}{4}}V_0, \quad p_t = \xi_t^{-\frac{1}{2}}P_0. \tag{5.04}$$

The viscosity and heat-conduction terms are retained if

$$\{(\gamma M^2)^\omega/R_L\delta^4\}\{\delta^6/\phi_t^2\theta_t^{1-\omega}\} = O(1).$$

That $\{(\gamma M^2)^\omega/R_L\delta^4\} \equiv \Lambda = O(1)$ was required in § 4. That $\{\delta^6/\phi_t^2\theta_t^{1-\omega}\} = O(1) \equiv 1$ remains to be demonstrated. However, if $\{\delta^6/\phi_t^2\theta_t^{1-\omega}\} = 1$, it should be noted that $\phi_t\theta_t/\delta^3 = \theta_t^{\frac{1}{2}(1+\omega)}\to 0$, which was postulated in deriving equation (5.03).

Now consider the possibility of a similar solution for the temperature in the transition layer. (It is not necessary to consider such a possibility for the tangential velocity u_t.) If the temperature has the form

$$T_t = \xi_t^{c_1}\Theta_t(\zeta_t), \quad \text{with} \quad \zeta_t = \psi_t/\xi_t^{c_2}, \tag{5.05}$$

then the energy equation may be written as

$$\frac{\Lambda P_0}{c_2\sigma}\frac{d}{d\zeta_t}\left(\frac{1}{\Theta_t^{1-\omega}}\frac{d\Theta_t}{d\zeta_t}\right)+\zeta_t\frac{d\Theta_t}{d\zeta_t}-\frac{1}{c_2}\left(c_1+\frac{\gamma-1}{2\gamma}\right)\Theta_t = 0, \tag{5.06}$$

subject to the restriction that, for $n = \frac{3}{4}$,

$$c_2 = \tfrac{1}{4}\{1-2(1-\omega)c_1\}. \tag{5.07}$$

First, the asymptotic behaviour of equation (5.06) as $\zeta_t\to\infty$ must be investigated to determine whether or not there is a possibility of matching the transition layer temperature to that of the inviscid shock layer. By inspection, it can be seen that the expansion

$$\Theta_t \sim K\zeta_t^{-q}[1+O(\zeta_t^{-\{2-q(1-\omega)\}})], \tag{5.08}$$

where $q = -\frac{1}{2}\{c_1+(\gamma-1)/2\gamma\}$ and is such that q and $\{2-q(1-\omega)\}$ are positive quantities, represents the solution of equation (5.06) as $\zeta_t \to \infty$. Thus, in terms of the stream function ψ_t,

$$T_t \sim K\xi_t^{c_1+qc_2}\psi_t^{-q} \quad \text{as} \quad \psi_t \to \infty. \tag{5.09}$$

But, from the inviscid shock layer (equation (3.13)) for $n = \frac{3}{4}$,

$$T_h \sim \Theta_0 \xi_h^{-(\gamma-1)/2\gamma}\psi_h^{-2/3\gamma} \quad \text{as} \quad \psi_h \to 0. \tag{5.10}$$

Therefore, from equations (5.09) and (5.10), it is clear that there is matching between the inviscid shock layer and the viscous transition layer, as $\psi_h \to 0$ and $\psi_t \to \infty$, if

$$\theta_t \phi_t^{2/3\gamma} = \delta^{2(3\gamma+1)/3\gamma}, \tag{5.11}$$

$$K = \Theta_0, \quad q = 2/3\gamma, \quad c_1 + (2c_2/3\gamma) = -(\gamma-1)/2\gamma. \tag{5.12}$$

Equations (5.07) and (5.12) thus require that the similarity constants, c_1 and c_2, be

$$c_1 = -\frac{\gamma-\frac{2}{3}}{2\gamma\{1-(1-\omega)/3\gamma\}}, \quad c_2 = \frac{1+(1-\omega)(\gamma-1)/\gamma}{4\{1-(1-\omega)/3\gamma\}}. \tag{5.13}$$

Also, since $q = 2/3\gamma$, equation (5.06) simplifies to

$$\frac{1}{\lambda}\frac{d}{d\zeta_t}\left(\frac{1}{\Theta_t^{1-\omega}}\frac{d\Theta_t}{d\zeta_t}\right) + \zeta_t\frac{d\Theta_t}{d\zeta_t} + \frac{2}{3\gamma}\Theta_t = 0; \quad \lambda = \frac{\sigma c_2}{\Lambda P_0}. \tag{5.14}$$

For $(1-\omega) > 0$, equation (5.14) can be recast as a first-order differential equation and can be studied more completely by the method of singular points. To do this, consider the new variables

$$F = \zeta_t^{-2\omega/(1-\omega)}\Theta_t^{-\omega}, \quad G = \omega\frac{\zeta_t}{\Theta_t}\frac{d\Theta_t}{d\zeta_t}. \tag{5.15}$$

The mapping equation for these variables is

$$\frac{d\zeta_t}{\zeta_t} = -\frac{dF}{F\{G+(2\omega)/(1-\omega)\}}, \tag{5.16}$$

and the fundamental equation in the (F, G)-plane is

$$\frac{dG}{dF} = -\frac{G(1-G)F^{(1-\omega)/\omega} - \lambda\{G+(2\omega)/(3\gamma)\}}{F^{1/\omega}\{G+(2\omega)/(1-\omega)\}}. \tag{5.17}$$

The isocline of zero slope of this equation is

$$F^{(1-\omega)/\omega} = \frac{\lambda\{G+(2\omega)/(3\gamma)\}}{G(1-G)}, \tag{5.18}$$

and the isoclines of infinite slope are the lines

$$F = 0; \quad G = -(2\omega)/(1-\omega). \tag{5.19}$$

Therefore, the two singular points of the equation, the intersections of the zero-slope isocline with the infinite-slope isoclines, are

$$(F_\infty, G_\infty) = (0, -2\omega/3\gamma);$$

$$(F_0, G_0) = \left(\left[\frac{1+\omega}{1-\omega}\frac{4\Lambda P_0/\sigma}{\{1+(1-\omega)(\gamma-1)/\gamma\}}\right]^{-\omega/(1-\omega)}, -\frac{2\omega}{1-\omega}\right). \tag{5.20}$$

Expressing the variables F and G as

$$F_\infty = f_\infty, \quad G = -(2\omega)/(1-\omega) - g_\infty, \tag{5.21}$$

equation (5.17) near the first singular point is approximately

$$\frac{dg_\infty}{df_\infty} - \left[\lambda\frac{1-\omega}{2\omega}\frac{1}{\{1-(1-\omega)/(3\gamma)\}}\right]\frac{1}{f_\infty^{1/\omega}}g_\infty = -\left[\frac{1-\omega}{3\gamma}\frac{\{1+(2\omega)/(3\gamma)\}}{\{1-(1-\omega)/(3\gamma)\}}\right]\frac{1}{f_\infty}. \tag{5.22}$$

The solution of this equation, with $g_\infty \to 0$ as $f_\infty \to 0$, is

$$g_\infty = \text{const.}\, e^{-Q} + g_\infty^* \sum_{m=0}^{\infty} m!\, Q^{-(1+m)},$$

$$\frac{dg_\infty}{df_\infty} = \left(\frac{1-\omega}{\omega}\right) A^{-\omega/(1-\omega)} Q^{1/(1-\omega)}\left[\text{const.}\, e^{-Q} + g_\infty^* \sum_{m=0}^{\infty}(1+m)!\, Q^{-(2+m)}\right], \tag{5.23a}$$

where

$$g_\infty^* = \left[\frac{\omega}{3\gamma}\left(\frac{1+(2\omega)/(3\gamma)}{1-(1-\omega)/(3\gamma)}\right)\right], \quad Q = \left[\frac{\frac{1}{2}\lambda}{\{1-(1-\omega)/(3\gamma)\}}\right] f_\infty^{-(1-\omega)/\omega} = A f_\infty^{-(1-\omega)/\omega}. \tag{5.23b}$$

In addition, near this singular point, the mapping equation yields

$$f_\infty \sim \zeta_t^{-2\omega\{1-(1-\omega)/(3\gamma)\}/(1-\omega)}, \quad \text{i.e. } \zeta_t \to \infty \quad \text{as} \quad f_\infty \to 0. \tag{5.24}$$

Thus, in terms of the original variables, the behaviour near this singular point is

$$\Theta_t \sim \zeta_t^{-2/3\gamma} \quad \text{as} \quad \zeta_t \to \infty, \tag{5.25}$$

the asymptotic behaviour already shown to be required for matching to the inviscid shock layer.

Taking F and G to be

$$F = \left[\frac{1+\omega}{1-\omega}\frac{4\Lambda P_0/\sigma}{\{1+(1-\omega)(\gamma-1)/\gamma\}}\right]^{-\omega/(1-\omega)} - f_0, \quad G = -\frac{2\omega}{1-\omega} + g_0, \tag{5.26}$$

the equation near the second singular point is approximately

$$\frac{dg_0}{df_0} - \left[\frac{1+\omega}{1-\omega}\frac{2}{F_0^2}\right]\frac{f_0}{g_0} = \left[\frac{2\omega}{1-\omega} - \frac{(1+\omega)/(3\gamma)}{\{1-(1-\omega)/(3\gamma)\}}\right]\frac{1}{F_0}. \tag{5.27}$$

The solution of this equation, with $g_0 \to 0$ as $f_0 \to 0$, is

$$g_0 = \left(\frac{g_0^*}{F_0}\right) f_0, \quad \frac{dg_0}{df_0} = \left(\frac{g_0^*}{F_0}\right), \tag{5.28a}$$

where

$$g_0^* = \frac{1}{2}\left(\left[\frac{2\omega}{1-\omega} - \frac{(1+\omega)/(3\gamma)}{\{1-(1-\omega)/(3\gamma)\}}\right] \pm \left\{\left[\frac{2\omega}{1-\omega} - \frac{(1+\omega)/(3\gamma)}{\{1-(1-\omega)/(3\gamma)\}}\right]^2 + 8\left(\frac{1+\omega}{1-\omega}\right)\right\}^{\frac{1}{2}}\right). \tag{5.28b}$$

Near this singular point, the mapping equation yields

$$f_0 \sim \zeta_t^{g_0^*}. \tag{5.29}$$

This means that: $\zeta_t \to 0$ as $f_0 \to 0$ if $g_0^* > 0$; $\zeta_t \to \infty$ as $f_0 \to 0$ if $g_0^* < 0$. In order that both singular points do not map into the same point in the physical plane, it is necessary that g_0^* be positive. Hence, the relevant root in equation (5.28) is the positive one. In terms of the original variables, the behaviour near this singular point becomes

$$\Theta_t \sim a_1 \zeta_t^{-2/(1-\omega)} + \ldots \quad \text{as} \quad \zeta_t \to 0, \tag{5.30}$$

or,

$$\left.\begin{aligned} T_t &\sim a_1 \xi_t^{c_1} (\psi_t / \xi_t^{c_2})^{-2/(1-\omega)} + \ldots, \\ &\sim a_1 \xi_t^{1/2(1-\omega)} \psi_t^{-2/(1-\omega)} + \ldots \quad \text{as} \quad \psi_t \to 0, \end{aligned}\right\} \tag{5.31}$$

where this a_1 is the quantity introduced in equation (4.12).

From a comparison of equations (4.13) and (5.31), it can be seen that the transition layer does match to the viscous boundary layer, as $\psi_t \to 0$ and $\psi_b \to \infty$, if

$$\theta_t \phi_t^{2/(1-\omega)} = (\delta^3)^{2/(1-\omega)} \quad \text{or} \quad (\delta^6/\phi_t^2 \theta_t^{1-\omega}) = 1. \tag{5.32}$$

This is exactly the relation that was required for the retention of the viscosity and heat-conduction terms in the transition layer equations. Solving equations (5.11) and (5.32), it is found that

$$\left.\begin{aligned} \phi_t &= \delta^{J_1}, \quad J_1 = \frac{3-(1-\omega)(3\gamma+1)/3\gamma}{1-(1-\omega)/3\gamma}, \\ \theta_t &= \delta^{J_2}, \quad J_2 = \frac{2(3\gamma-2)/3\gamma}{1-(1-\omega)/3\gamma}. \end{aligned}\right\} \tag{5.33}$$

It is easily verified that $\delta^3 \ll \phi_t \ll \delta$ and $\delta^2 \ll \theta_t \ll 1$, as was postulated in the formulation of the transition layer. Finally, the thickness of the transition layer is determined to be of $O(\delta^{J_1+J_2-2}) \ll \delta$.

In the above demonstration, that the postulated transition layer matches to the inviscid shock layer as $\zeta_t \to \infty$ and matches to the viscous boundary layer as $\zeta_t \to 0$, it has been tacitly assumed that the existence of solutions for the transition layer temperature, etc., between these limits can be proved. Since the solutions of the viscous boundary layer do not depend on complete solutions for the flow quantities in the transition layer, no attempt at such (numerical) solutions has been made. However, it is possible to show that a solution for the temperature exists from a study of the phase plane between the points which correspond to $\zeta_t \to 0$ and $\zeta_t \to \infty$, (F_0, G_0) and (F_∞, G_∞), respectively. The proof is presented for $\frac{1}{2} < \omega < 1$. The proof for $\omega = \frac{1}{2}$ is not considered here.

The isoclines of infinite slope are the lines

$$F = F_\infty = 0 \quad \text{and} \quad G = G_0 = -(2\omega)/(1-\omega).$$

They are denoted as curves A and B, respectively, in figure 2. The expression for the isocline of zero slope is given by equation (5.18). As far as the region of interest is concerned, the isocline of zero slope: (i) starts at (F_∞, G_∞) with a slope of $(-\infty)$; (ii) proceeds with a negative slope until the turning point (F_r, G_r), where $F_r > F_0 > F_\infty$, $G_0 < G_r < G_\infty$, is reached; and (iii) continues from (F_r, G_r), with a positive slope to $(0, -\infty)$, passing through (F_0, G_0). The portion of the isocline of zero slope in the region where $G > G_0$ is denoted by C in figure 2. Further, from an examination of equation (5.17), it has been determined that, in the region bounded by A, B, and C, the shaded region R in figure 2, the slope, dG/dF, is negative.

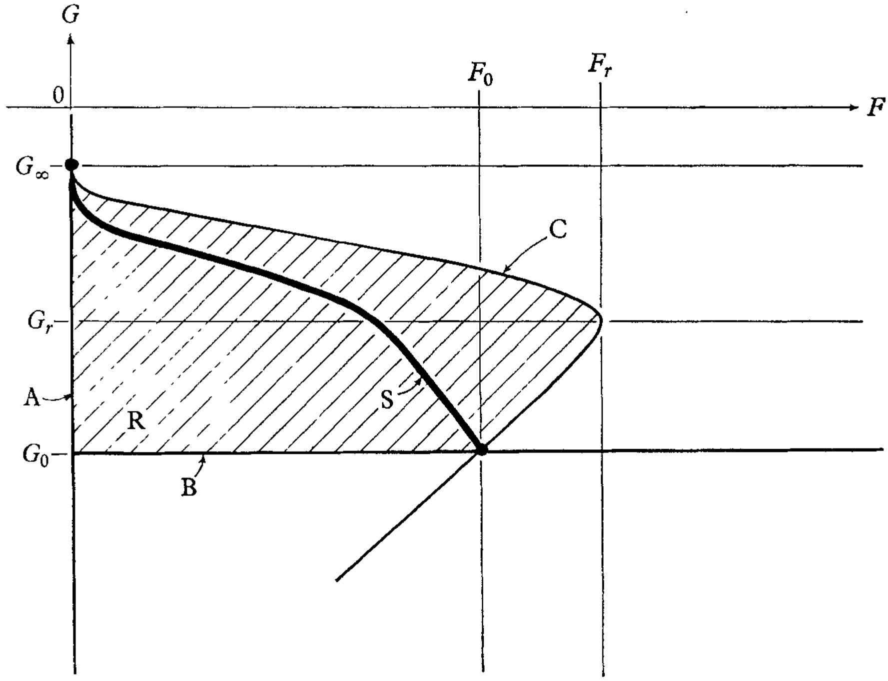

FIGURE 2. Schematic diagram of the phase plane temperature solution for the viscous transition layer.

Therefore, the trajectory, which enters region R at (F_0, G_0) with a slope of $dG/dF = (-g_0^*/F_0) < 0$, due to the nature of the region and its bounding curves, is able to leave R only at the point (F_∞, G_∞) with a slope of $dG/dF = -\infty$. Such a trajectory is the phase plane solution curve for the transition layer temperature. It is shown as curve S in figure 2.

6. Conclusions

In §3, it has been shown that the temperature near the inner edge of the inviscid shock layer is

$$T_1/\gamma M^2 T_\infty \sim \delta^2 \Theta_0 \xi^{-(\gamma-1)/2\gamma} (\psi/\delta)^{-2/3\gamma}, \quad (\psi/\delta) \to 0. \tag{6.01}$$

In §4, the temperature at the outer edge of the viscous boundary layer, for $\mu_1/\mu_\infty = (T_1/T_\infty)^\omega$, $(1-\omega) > 0$, has been shown to have the form

$$T_1/\gamma M^2 T_\infty \sim a_1 \zeta_b^{-2/(1-\omega)}, \quad \zeta_b = (\psi/\delta^3 \xi^{\frac{1}{2}}) \to \infty. \tag{6.02}$$

From the above expressions, it is clear that the viscous boundary-layer solution cannot match directly to the inviscid shock-layer solution. In §5, a viscous transition layer is introduced, in which the temperature has the form

$$T_1/\gamma M^2 T_\infty \sim \delta^{J_2} \xi^{c_1} \Theta_t(\psi/\delta^{J_1} \xi^{c_2}). \tag{6.03}$$

It has been demonstrated that this temperature distribution matches with that of the inviscid shock layer at the transition layer's outer edge and with that of the viscous boundary layer at the transition layer's inner edge.

With the above as background, consider the solution for $\omega = 1$. For $\omega = 1$, the temperature at the outer edge of the viscous boundary layer has been shown (Ladyzhenskii 1963) to have the form

$$T_1/\gamma M^2 T_\infty \sim C_3 \zeta_b^{-C_1} \exp(-C_2 \zeta_b^2), \quad \zeta_b \to \infty \quad \text{with} \quad C_1, C_2, C_3 = \text{consts.} \qquad (6.04)$$

Then, from a comparison of equations (6.01) and (6.04), it follows that, for $\omega = 1$ also, the direct matching of the viscous boundary layer and the inviscid shock layer is not possible.

At first glance, it would seem possible to make the matching complete for $\omega = 1$ by the introduction of a transition layer analogous to the one for $(1-\omega) > 0$. This is not possible because, for $\omega = 1$, the temperature at the outer edge of the boundary layer goes to zero exponentially, rather than algebraically, and there is no way to match directly to exponential decay. Oguchi (1958) claims that this problem can be overcome by a matching of the zeroth- and first-order boundary-layer approximations for the temperature to the zeroth-order shock-layer approximation. However, the author feels that the strong-interaction problem for $\omega = 1$ still represents an area for further investigation.

Appendix

Solutions for the viscous boundary layer

The similarity continuity, momentum, and energy equations for the viscous boundary layer are given in equation (4.09), and the boundary conditions for these equations are given in equation (4.10).

The solutions of these equations have been found by Dewey (1963) in terms of different variables. To be able to interpret his results, consider a change of variables from those employed in this paper (ζ_b, U_b, Θ_b) to variables similar to those employed by Dewey (η, f, H), where

$$\left.\begin{aligned} \eta &= [(\tfrac{1}{2}\Gamma H_w)^{1-\omega}(1/4\Lambda P_0)]^{\frac{1}{2}} \int_0^{\zeta_b} \frac{dt}{U_b(t)}, \\ f &= [(\tfrac{1}{2}\Gamma H_w)^{1-\omega}(1/4\Lambda P_0)]^{\frac{1}{2}} \zeta_b, \quad H = U_b^2 + (2/\Gamma)\,\Theta_b, \end{aligned}\right\} \qquad (\text{A } 01)$$

where $\Gamma = (\gamma-1)/\gamma$ and $H_w = (2\Theta_{b,w}/\Gamma)$. In terms of these new variables, the momentum and energy equations become

$$\left.\begin{aligned} \frac{d}{d\eta}\left(N\frac{d^2f}{d\eta^2}\right) + f\frac{d^2f}{d\eta^2} + \Gamma\left[H - \left(\frac{df}{d\eta}\right)^2\right] &= 0, \\ \frac{d}{d\eta}\left(N\frac{d}{d\eta}\left\{\frac{H}{\sigma} - \left(\frac{1-\sigma}{\sigma}\right)\left(\frac{df}{d\eta}\right)^2\right\}\right) + f\frac{dH}{d\eta} &= 0, \end{aligned}\right\} \qquad (\text{A } 02)$$

where $N = \{H_w/[H-(df/d\eta)^2]\}^{1-\omega}$. Since

$$U_b = (df/d\eta) \quad \text{and} \quad \Theta_b = \tfrac{1}{2}\Gamma[H-(df/d\eta)^2],$$

the boundary conditions for equation (A 02) are

$$\left.\begin{array}{l} H \to H_w, \quad (df/d\eta), \quad f \to 0, \quad \text{as} \quad \eta \to 0, \\ H \to 1, \quad (df/d\eta) \to 1, \quad \text{as} \quad \eta \to \infty. \end{array}\right\} \tag{A 03}$$

The wall friction and heat transfer become

$$\left.\begin{array}{l} \dfrac{[\mu_1(\partial u_1/\partial y_1)]_w}{\rho_\infty u_\infty^2} = \dfrac{1}{2}\left(\dfrac{3\Gamma}{4V_0}\right)^{\frac{1}{2}}\left[\left(\dfrac{2}{\Gamma H_w}\right)^{1-\omega}\dfrac{P_0^{\frac{1}{3}}(\gamma M^2)^{\omega}}{R_L x}\right]^{\frac{3}{4}} I^{\frac{1}{2}}\left(\dfrac{d^2f}{d\eta^2}\right)_w, \\ \dfrac{[k_1(\partial T_1/\partial y_1)]_w}{\rho_\infty u_\infty^3} = \dfrac{1}{4\sigma}\left(\dfrac{3\Gamma}{4V_0}\right)^{\frac{1}{2}}\left[\left(\dfrac{2}{\Gamma H_w}\right)^{1-\omega}\dfrac{P_0^{\frac{1}{3}}(\gamma M^2)^{\omega}}{R_L x}\right]^{\frac{3}{4}} I^{\frac{1}{2}}\left(\dfrac{dH}{d\eta}\right)_w, \end{array}\right\} \tag{A 04}$$

where

$$I = \int_0^\infty [H - (df/d\eta)^2]\, d\eta. \tag{A 05}$$

To obtain these expressions, the continuity equation is used. In using this equation, it is found that, in order to satisfy the boundary conditions, $V_b \to 0$ as $\eta \to 0$, $V_b \to V_0$ as $\eta \to \infty$, the quantity Λ must be

$$\Lambda = (\tfrac{1}{2}\Gamma H_w)^{1-\omega} P_0 (4V_0/3\Gamma I)^2. \tag{A 06}$$

The values of $(d^2f/d\eta^2)_w$, $(dH/d\eta)_w/(1-H_w)$, and I for $\gamma = 1{\cdot}4$ and different values of ω, σ, and H_w are given in Dewey's paper. From equation (A 04) and Dewey's values, it is clear that the greatest effect on the wall friction and heat transfer, due to ω being less than unity, comes from the $M^{\frac{3}{2}\omega}$-term, since the variation in the other terms is relatively negligible.

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